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A NEW TECHNIQUE FOR THE CALCULATION OF EFFECTIVE MESONIC POTENTIAL AT FINITE TEMPERATURE IN THE LOGARITHMIC QUARK-SIGMA MODEL

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ABSTRACT. The logarithmic sigma model describes the interactions between quarks via sigma and pion exchanges. The effective mesonic potential is extended to the finite temperature and it is numerically calculated using n-midpoint rule. Meson properties such as the phase transition, the sigma and pion masses, and the critical point temperature are examined as functions of temperature. The obtained results are compared with other approaches. We conclude that the calculated effective potential is successfully to predict the meson properties.

1. INTRODUCTION

The study of matter at very high temperature and densities is of interest because of its relevance to particle physics and astrophysics. According to the standard big bang model, it is believed that a series of phase transitions happened at the early stages of the evolution of universe. The QCD phase transition being one of them. The lattice QCD and effective field theories are two main approaches to calculate the phase transition and meson properties at finite temperature [1]. This subject has been under intense theoretical study using various effective field theory models, such as the Nambu–Jona-Lasinio Model [2-4], a linear sigma model [5-8]. One of the effective models in describing baryon properties is the linear sigma model, which was suggested earlier by Gell-Mann and Levy [9] to describe the nucleons interacting via sigma (σ) and pion (π) exchanges. At finite temperature, the model gives a good description of the phase transition by using the Hartree approximation [1, 10-12] within the Cornwall–Jackiw–Tomboulis (CJT) formalism [13]. However, there exists serious difficulty concerning the renormalization of the CJT effective action in the Hartree approximation. Baacke and Michalski [14] indicated that the phase transition can be obtained beyond the large N and Hartree

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approximation through the systematic expansions are bases on the resummation scheme by Cornwall-Jackiw-Tombonlis and 2PI scheme.

The aim of this work is to calculate the effective logarithmic mesonic potential, the phase transition, and meson masses at finite temperature. The logarithmic mesonic potential at zero temperature was suggested in Ref. [15] to provide a good description of hadron properties. The used method is different as a new technique in comparison with other works as in Refs. [5-8]. In addition, the pressure is investigated as a function temperature.

This paper is organized as follows: In Sec. 2, the linear sigma model at zero temperature and finite temperature are explained briefly. The numerical calculations and discussion of the results are presented in Sec. 3. Summary and conclusion is presented in Sec. 4.

2. THE LOGARITHMIC QUARK-SIGMA MODEL

2.1. The Logarithmic Potential at zero Temperature. The Lagrangian density of quark sigma model that describes the interactions between quarks via the σ - and π -meson exchange [15]. The Lagrangian density is,

$$L(r) = i\bar{\Psi}\partial_\mu\gamma^\mu\Psi + \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi}) + g\bar{\Psi}(\sigma + i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})\Psi - U^{T(0)}(\sigma, \boldsymbol{\pi}), \quad (1)$$

with

$$U^{T(0)}(\sigma, \boldsymbol{\pi}) = \lambda_1^2(\sigma^2 + \boldsymbol{\pi}^2) - \lambda_2^2\log(\sigma^2 + \boldsymbol{\pi}^2) + m_\pi^2 f_\pi \sigma, \quad (2)$$

$U^{T(0)}(\sigma, \boldsymbol{\pi})$ is the meson-meson interaction potential where Ψ, σ and $\boldsymbol{\pi}$ are the quark, sigma, and pion fields, respectively. In the mean-field approximation the meson fields are treated as time-independent classical fields. This means that we replace power and products of the meson fields by corresponding powers and products of their expectation values. The meson-meson interactions in Eq. (2) lead to hidden chiral $SU(2) \times SU(2)$ symmetry with $\sigma(r)$ taking on a vacuum expectation value

$$\langle\sigma\rangle = -f_\pi, \quad (3)$$

where $f_\pi = 93$ MeV is the pion decay constant. The final term in Eq. (2) is included to break the chiral symmetry. It leads to partial conservation of axial-vector isospin current (PCAC). The parameters λ^2, ν^2 can be expressed in terms of f_π , the masses of mesons as,

$$\lambda_1^2 = \frac{1}{4}(m_\sigma^2 + m_\pi^2), \quad (4)$$

$$\lambda_2^2 = \frac{f_\pi^2}{4}(m_\sigma^2 - m_\pi^2). \quad (5)$$

2.2. The Effective Mesonic Potential at Finite-Temperature

. In Eq. (2), The effective potential is extended to calculate the chiral interacting mesons with quarks at finite temperature. The quarks are considered a heat bath in local thermal equilibrium

$$U_{eff}(\sigma, \boldsymbol{\pi}, T) = U^{T(0)}(\sigma, \boldsymbol{\pi}) - 24T \int \frac{d^3p}{(2\pi)^3} \ln(1 + e^{\frac{-\sqrt{p^2 + g^2(\sigma^2 + \pi^2)}}{T}}), \quad (6)$$

the first term is the potential in the tree level defines in Eq. (2) and the second term is for the chiral meson fields interacts with quarks at finite temperature and zero-chemical potential. Non-zero values of the chiral fields in the chiral broken

phase dynamically generate a quark mass $m_q = gf_\pi$. The integration is taken over momentum volume (for details, see Ref. [16]).

3. NUMERICAL CALCULATIONS AND DISCUSSION OF RESULTS

3.1. Numerical Calculations. The purpose of this section is to calculate the effective mesonic potential σ and π - masses, and the pressure. We rewrite Eq. (6) as follows

$$U_{eff}(\sigma, \pi, T) = U^{T(0)}(\sigma, \pi) - \frac{12T}{\pi^2} \int_0^\infty p^2 dp (\ln(1 + e^{-\frac{\sqrt{p^2 + g^2(\sigma^2 + \pi^2)}}{T}})). \quad (7)$$

Eq. (7) is written in the dimensionless form as follows

$$U_{eff}(\sigma, \pi, T) = f_\pi^4 [U^{T(0)}(\sigma', \pi') - \frac{12T'}{\pi^2} \int_0^\infty p'^2 (\ln e^{(-\frac{1}{T'} \sqrt{p'^2 + g^2(\sigma'^2 + \pi'^2)})} + 1) dp'], \quad (8)$$

where

$$U^{T(0)}(\sigma', \pi') = \lambda_1'^2 (\sigma'^2 + \pi'^2) - \lambda_2'^2 \log f_\pi^2 (\sigma'^2 + \pi'^2) + m_\pi'^2 \sigma', \quad (9)$$

and

$$\lambda_1'^2 = \frac{1}{4} (m_\sigma'^2 + m_\pi'^2), \lambda_2'^2 = \frac{1}{4} (m_\sigma'^2 - m_\pi'^2), \quad (10)$$

where $\sigma', \pi', T', m_\sigma',$ and m_π' are in unit of f_π . Therefore $U^{T(0)}(\sigma', \pi')$ is the dimensionless form of $U^{T(0)}(\sigma, \pi)$. Substituting $p' = -\ln y$ into Eq. (8), we obtain:

$$U_{eff}(\sigma, \pi, T) = f_\pi^4 [U^{T(0)}(\sigma', \pi') - \frac{12T'}{\pi^2} \int_0^1 \frac{(\ln y)^2}{y} \ln \left(\exp \left(-\frac{1}{T'} \sqrt{(\ln y)^2 + g^2(\sigma'^2 + \pi'^2)} \right) + 1 \right) dy],$$

hence we can write the dimensionless form of $U_{eff}(\sigma, \pi, T)$ as follows:

$$U_{eff}(\sigma', \pi', T') = [U^{T(0)}(\sigma', \pi') - \frac{12T'}{\pi^2} \int_0^1 \frac{(\ln y)^2}{y} \ln \left(\exp \left(-\frac{1}{T'} \sqrt{(\ln y)^2 + g^2(\sigma'^2 + \pi'^2)} \right) + 1 \right) dy]. \quad (11)$$

By using midpoint rule, we obtain the approximate integral as follows:

$$U_{eff}(\sigma, \pi, T) = f_\pi^4 [U^{T(0)}(\sigma', \pi') - \frac{12T'}{\pi^2} A \ln \left(\exp \left(-\frac{1}{T'} \sqrt{g^2(\sigma'^2 + \pi'^2) + B} \right) + 1 \right)], \quad (12)$$

where

$$A = \frac{1}{n} \sum_{i=0}^n \frac{1}{\frac{1}{n}i + \frac{1}{2n}} \ln^2 \left(\frac{1}{n}i + \frac{1}{2n} \right), \quad B = \sum_{i=0}^n \ln^2 \left(\frac{1}{n}i + \frac{1}{2n} \right). \quad (13)$$

(for details, see Refs. [17, 18]). In Ref. [19], the authors applied the second derivation of the effective potential respect to σ' and π' to obtain the effective meson masses. Then, the first derivative of the effective potential $U_{eff}(\sigma', \pi', T')$ is given by

$$\frac{\partial U_{eff}(\sigma', \pi', T)}{\partial \sigma} = \frac{1}{f_\pi} \frac{\partial U_{eff}(\sigma', \pi', T)}{\partial \sigma'} = f_\pi^3 \left[\frac{\partial U_0(\sigma', \pi', T)}{\partial \sigma'} - \frac{12T'}{\pi^2} \frac{d_1 d_2}{d_3 + d_3 d_2} \right], \quad (14)$$

where

$$\begin{aligned} d_1 &= -Ag^2\sigma', \quad d_2 = \exp\left(-\frac{1}{T'}\sqrt{B+g^2(\sigma'^2+\pi'^2)}\right), \\ d_3 &= T'\sqrt{B+g^2(\sigma'^2+\pi'^2)}. \end{aligned} \quad (15)$$

Then, we obtain the effective sigma mass as follows

$$m_\sigma^2(T) = \frac{\partial^2 U_{eff}(\sigma', \pi', T)}{f_\pi^2 \partial \sigma'^2} = f_\pi^2 \left[\frac{\partial^2 U_0(\sigma', \pi', T)}{\partial \sigma'^2} - \frac{12T'}{\pi^2} \frac{\partial}{\partial \sigma'} \left(\frac{d_1 d_2}{d_3 + d_3 d_2} \right) \right], \quad (16)$$

$$m_\sigma(T) = f_\pi \left[\frac{\partial^2 U_0(\sigma', \pi', T)}{\partial \sigma'^2} - \frac{12T'}{\pi^2} \frac{\partial}{\partial \sigma'} \left(\frac{d_1 d_2}{d_3 + d_3 d_2} \right) \right]^{\frac{1}{2}}, \quad (17)$$

where

$$\begin{aligned} \frac{\partial}{\partial \sigma'} \left(\frac{d_1 d_2}{d_3 + d_3 d_2} \right) &= d_1 \frac{d'_2}{d_3 + d_2 d_3} + d_2 \frac{d'_1}{d_3 + d_2 d_3} - \\ &\quad d_1 \frac{d_2}{(d_3 + d_2 d_3)^2} (d'_3 + d_2 d'_3 + d_3 d'_2), \end{aligned} \quad (18)$$

with

$$d'_1 = \frac{\partial d_1}{\partial \sigma'} = -Ag^2, \quad (19)$$

$$d'_2 = \frac{\partial d_2}{\partial \sigma'} = -\frac{1}{T'} g^2 \frac{\sigma'}{\sqrt{B+g^2(\sigma'^2+\pi'^2)}} \exp\left(-\frac{1}{T'}\sqrt{B+g^2(\sigma'^2+\pi'^2)}\right), \quad (20)$$

$$d'_3 = \frac{\partial d_3}{\partial \sigma'} = T g^2 \frac{\sigma'}{\sqrt{B+g^2(\sigma'^2+\pi'^2)}}. \quad (21)$$

Similarly, we obtain the effective pion mass as follows

$$m_\pi(T) = f_\pi \left[\frac{\partial^2 U(\sigma', \pi')}{\partial \pi'^2} - \frac{12}{\pi^2} T' \frac{\partial}{\partial \pi'} \left(\frac{d_4 d_2}{d_3 + d_3 d_2} \right) \right]^{\frac{1}{2}}, \quad (22)$$

where

$$d_4 = -Ag^2\pi', \quad (23)$$

$$\begin{aligned} \frac{\partial}{\partial \pi'} \left(\frac{d_4 d_2}{d_3 + d_3 d_2} \right) &= d_2 \frac{d'_4}{d_3 + d_2 d_3} + d_4 \frac{d'_2}{d_3 + d_2 d_3} - \\ &\quad d_2 \frac{d_4}{(d_3 + d_2 d_3)^2} (d'_3 + d_2 d'_3 + d_3 d'_2), \end{aligned} \quad (24)$$

where

$$d'_4 = \frac{\partial d_4}{\partial \pi'} = -Ag^2, \quad (25)$$

$$d'_2 = \frac{\partial d_2}{\partial \pi'} = -\frac{1}{T} g^2 \frac{\pi'}{\sqrt{B+g^2(\sigma'^2+\pi'^2)}} \exp\left(-\frac{1}{T}\sqrt{B+g^2(\sigma'^2+\pi'^2)}\right), \quad (26)$$

$$d'_3 = \frac{\partial d_3}{\partial \pi'} = T g^2 \frac{\pi'}{\sqrt{B+g^2(\sigma'^2+\pi'^2)}}. \quad (27)$$

In Ref. [16], the pressure is given in the dimensionless form as follow:

$$P'(\sigma', \pi', T) = U^{T(0)}(\sigma', \pi') - U_{eff}(\sigma', \pi', \mathbf{T}'), \quad (28)$$

4. RESULTS AND DISCUSSION

In Eq. (7), The integration is solved by the n-midpoint algorithm and the index n is taken $n = 1000$ to get a good accuracy for a numerically integration. In Ref. [20], the authors used a different method for calculating the integration in the effective potential and they obtained it as a series of $M^2 = m^2 + \frac{\lambda}{2}\phi^2$, where the m, λ are parameters of the model. The difficulty in this potential is to determine a critical temperature T_c . The two terms were taken only in the expression of the potential since the increase of terms more than two terms will be the critical point is a complex value, leading T_c is not a physics quantity. In Refs. [1-4], authors used double bubble graphs instead of summing infinite set of daisy and superdaisy graphs using the tree level propagators. Therefore, the Feynman diagrams are needed. In the present work, The n-midpoint rule is used to avoid the difficulties in the above approaches. Hence, we did not need to apply Feynman diagrams. The parameters of the model such as $m_\pi = 140$ MeV, $m_\sigma = 600$ MeV, $f_\pi = 93$ MeV, and coupling constant g at zero-temperature are used as the initial parameters at the finite temperature. In addition, the effective pion and sigma masses are obtained as a second derivative respect to meson field. This method is used extensively in other works such as in the Ref. [19]. In this section, we examine the meson properties and the phase transition which depend on the calculation of the effective mesonic potential at finite temperature. We replaced the normal potential in the linear sigma model [5] at zero temperature by logarithmic mesonic potential [15]. In our previous work [15]. The logarithmic potential was successfully to predict hadron properties at zero temperature.

In Figs. (1, 2, 3), we investigate the behavior of the sigma and pion masses as functions of temperature. Moreover, the effect of the sigma mass and the coupling constant g on a critical point temperature in the presence of the explicit symmetry breaking ($m_\pi \neq 0$) and the chiral limit ($m_\pi = 0$) is investigated. In Fig. 1, the sigma and pion masses are plotted as functions of temperature at the presence of the explicit symmetry breaking term. The sigma mass decreases with increasing temperature and the pion mass increases with increasing temperature. The two curves crossed at a critical point temperature T_c where the sigma and pion masses have the same massive value. At $m_\sigma = 600$ MeV, we find the critical point temperature $T_c = 233$ MeV. By increasing the sigma mass up to $m_\sigma = 700$ MeV leads to $T_c = 309$ MeV. In comparison with lattice QCD results [19], the critical point temperature is found in the range $T_c = 100$ to 300 MeV. Therefore the present result is in agreement with lattice QCD results. Nemoto et al. [19] calculated the critical point temperature equal 230 MeV using the original sigma model. Abushady [5] calculated the critical point temperature equal 226 MeV in the original sigma model. Therefore, the present result is in good agreement with Refs. [5, 19]. In Fig. 2, the sigma and pion masses are plotted as functions of temperature in the chiral limit ($m_\pi = 0$). A similar behavior is obtained as in the case of the explicit symmetry term expect the pion mass equal zero at zero temperature. In comparison with original sigma model as in Refs. [1, 5, 19], we found that the behavior of the sigma and pion masses are in good agreement with Refs. [1, 5, 19]. In Fig. 3, we examine the effect of coupling constant g on the critical point temperature. We find that an increase on the coupling constant g leads to decrease in the critical point temperature $g = 3.76$ to $g = 4.48$ corresponding to $T_c = 243$ MeV to $T_c = 191$ MeV, respectively. In Fig. (4), the effective potential is plotted as a function of

temperature. We note that the potential decreases with increasing temperature. In order to get more insight into the nature of the phase transition and verify that the order phase transition is a second-order phase transition. We calculate the effective potential $U_{eff}(\Phi, T)$ as a function of phase transition ($\Phi = \sqrt{\sigma^2 + \pi^2}$) as seen in Fig. (4). The shape of the potential confirms that the phase transition is the second-order. Since it exhibits can degenerate one minima at $\Phi \neq 0$. The indication of the second-order phase transition has been reported in many works [14, 21- 23]. Also, We note the strong increase in the temperature T which unchanged the shape of the potential

Next, we need to examine the effect of finite temperature on the behavior of pressure where the quarks takes as a heat bath. In Fig. (5), the pressure is plotted as a function of temperature. The pressure increases with increasing temperature. Also, we note that the pressure value at lower-values of temperature is not sensitive in comparison with the pressure value at the higher-values of temperature. Hence the effect of largest values of temperature is more affected on the value of pressure. Berger and Christov [24] found that the pressure increases with increasing temperature in hot medium using the NJL model in the mean field approximation. The present behavior is in agreement with Ref. [24].

5. SUMMARY AND CONCLUSION

In this work, the effective mesonic potential is calculated by using n-midpoint algorithm in the logarithmic quark model. The meson properties, the phase transition, and the pressure are calculated using the effective mesonic potential. We summarized the following points: The behavior of sigma and pion masses as functions of temperature are investigated. A comparison with original sigma model is presented. The increase of sigma mass (m_σ) leads to increase the critical point temperature. The increase of the coupling constant (g) leads to decrease the critical point temperature. The phase transition is predicted as a second-order phase transition which agrees with other works. The critical point temperature and the pressure are calculated and are in agreement with other works. Therefore, the calculated effective logarithmic potential is successful to predict the meson masses, the phase transition, and the pressure at finite temperature.

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